Strings

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Today we're going to cover

- String matching
    - Naive algorithm
    - Knuth–Morris–Pratt (KMP) algorithm
- Tries
- Suffix tries
- Suffix trees
- Suffix arrays

String problems

- Strings frequently appear in our kind of problems
  - Reading input
  - Writing output
  - Parsing
  - Identifiers/names
  - Data
- But sometimes strings play the key role
  - We want to find properties of some given strings
  - Is the string a palindrome?
- Here we're going to talk about things related to the latter type of problems
- These problems can be hard, because the length of the strings are often huge

String matching

- Given a string S of length n,
- and a string T of length m,
- find all occurrences of T in S

- Note:
    - Occurrences may overlap
    - Assume strings contain characters from a constant-sized alphabet

String matching

Example:

- $S = \text{cabcababacaba}$
- $T = \text{aba}$

String matching

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- $S = \mathrm{cabcababacaba}$
- $T = \mathrm{aba}$

- Three occurrences:

String matching

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- $S = \text{cabcababacaba}$
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- Three occurrences:
  - cabcababacaba

String matching

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- $S = \text{cabcababacaba}$
- $T = \text{aba}$

- Three occurrences:
  - cabcababacaba
  - cabcababacaba

String matching

Example:

- $S = \mathrm{cabcababacaba}$
- $T = \mathrm{aba}$

- Three occurrences:
  - $\mathrm{cabc}\textcolor{red}{\mathrm{aba}}\mathrm{bacaba}$
  - $\mathrm{cabcab}\textcolor{red}{\mathrm{aba}}\mathrm{caba}$
  - $\mathrm{cabcababac}\textcolor{red}{\mathrm{aba}}$

Naive string matching algorithm

- For each substring of length m in S,
- check if that substring is equal to T.

Naive string matching algorithm

- S: `bacbababaabcbab`
- T: `ababaca`

Naive string matching algorithm

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Naive string matching algorithm

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Naive string matching algorithm

- S: `bacbababaabcbab`
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Naive string matching algorithm

- S: `bacbababaabcbab`
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Naive string matching algorithm

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Naive string matching algorithm

- S: `bacbababaabcbab`
- T: `ababaca`

Naive string matching algorithm

- S: `bacbababaabcbab`
- T: `         ababaca`

Naive string matching algorithm

```
int string_match(const string &s, const string &t) {
    int n = s.size(),
        m = t.size();

    for (int i = 0; i + m - 1 < n; i++) {
        bool found = true;
        for (int j = 0; j < m; j++) {
            if (s[i + j] != t[j]) {
                found = false;
                break;
            }
        }
        if (found) {
            return i;
        }
    }

    return -1;
}
```

Naive string matching algorithm

- Double for-loop
  - outer loop is $O(n)$ iterations
  - inner loop is $O(m)$ iterations worst case
- Time complexity is $O(nm)$ worst case

Naive string matching algorithm

- Double for-loop
  - outer loop is $O(n)$ iterations
  - inner loop is $O(m)$ iterations worst case
- Time complexity is $O(nm)$ worst case
- Can we do better?

Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
    - S: `bacbababaabcbab`
    - T: `ababaca`

Knuth–Morris–Pratt algorithm

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  - S: `bacbababaabcbab`
  - T: `ababaca`

Knuth–Morris–Pratt algorithm

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  - S: `bacbababaabcbab`
  - T: `   ababaca`

Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - S: `bacbababaabcbab`
  - T: `    ababaca`

Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - *S*: `bacbababaabcbab`
  - *T*: `    ababaca`

Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - S: `bacbababaabcbab`
  - T: `      ababaca`

Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - S: `bacbababaabcbab`
  - T: `ababaca`

Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - S: `bacbababaabcbab`
  - T: `          ababaca`
- The number of shifts depend on which characters are currently matched

Knuth–Morris–Pratt algorithm

- How are the number of shifts determined?
- Let $\pi[q] = \max\{k : k < q \text{ and } T[1 \dots k] \text{ is a suffix of } T[1 \dots q]\}$

Knuth–Morris–Pratt algorithm

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- Let $\pi[q] = \max\{k : k < q \text{ and } T[1 \ldots k] \text{ is a suffix of } T[1 \ldots q]\}$
- Example:

i	1	2	3	4	5	6	7
$T[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

Knuth–Morris–Pratt algorithm

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i	1	2	3	4	5	6	7
$T[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

- If, at position i, q characters match (i.e. $T[1 \ldots q] = S[i \ldots i + q - 1]$), then
  - if $q = 0$, shift pattern 1 position right
  - otherwise, shift pattern $q - \pi[q]$ positions right

Knuth–Morris–Pratt algorithm

- Example:
  - S: `bacbababaabcbab`
  - T: `ababaca`

Knuth–Morris–Pratt algorithm

- Example:
  - S: `bacbababaabcbab`
  - T: `    ababaca`
  - 5 characters match, so $q = 5$

Knuth–Morris–Pratt algorithm

- Example:
  - S: `bacbababaabcbab`
  - T: `ababaca`
  - 5 characters match, so $q = 5$
  - $\pi[q] = \pi[5] = 3$

Knuth–Morris–Pratt algorithm

- Example:
  - S: `bacbababaabcbab`
  - T: `    ababaca`
  - 5 characters match, so $q = 5$
  - $\pi[q] = \pi[5] = 3$
  - Then shift $q - \pi[q] = 5 - 3 = 2$ positions

Knuth–Morris–Pratt algorithm

- Example:
  - S: `bacbababaabcbab`
  - T: `ababaca`
  - 5 characters match, so $q = 5$
  - $\pi[q] = \pi[5] = 3$
  - Then shift $q - \pi[q] = 5 - 3 = 2$ positions
  - S: `bacbababaabcbab`
  - T: `ababaca`

Knuth–Morris–Pratt algorithm

- Given π, matching only takes $O(n)$ time
- π can be computed in $O(m)$ time
- Total time complexity of KMP therefore $O(n + m)$ worst case

Knuth–Morris–Pratt algorithm

```
int* compute_pi(const string &t) {

    int m = t.size();
    int *pi = new int[m + 1];
    if (0 <= m) pi[0] = 0;
    if (1 <= m) pi[1] = 0;
    for (int i = 2; i <= m; i++) {
        for (int j = pi[i - 1]; ; j = pi[j]) {
            if (t[j] == t[i - 1]) {
                pi[i] = j + 1;
                break;
            }
            if (j == 0) {
                pi[i] = 0;
                break;
            }
        }
    }

    return pi;
}
```

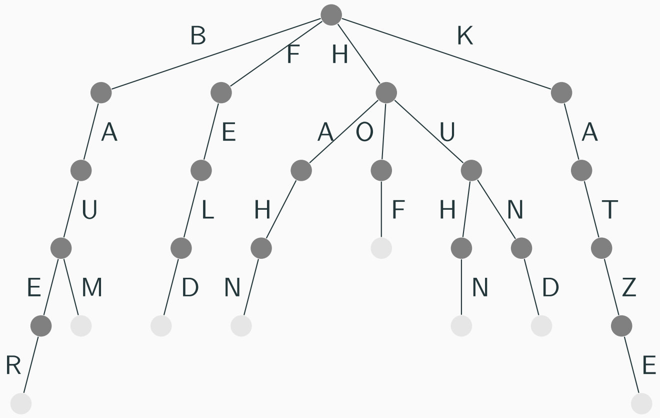
Knuth–Morris–Pratt algorithm

```
int string_match(const string &s, const string &t) {

    int n = s.size(),
        m = t.size();

    int *pi = compute_pi(t);

    for (int i = 0, j = 0; i < n; ) {
        if (s[i] == t[j]) {
            i++; j++;
            if (j == m) {
                return i - m;
            }
        }
        else if (j > 0) j = pi[j];
        else i++;
    }

    delete[] pi;
    return -1;
}
```

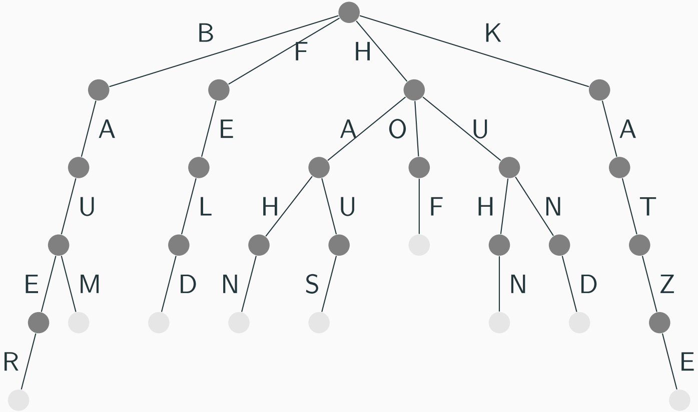
Sets of strings

- We often have sets (or maps) of strings
- Insertions and lookups usually guarantee $O(\log n)$ comparisons
- But string comparisions are actually pretty expensive...
- There are other data structures, like tries, which do this in a more clever way

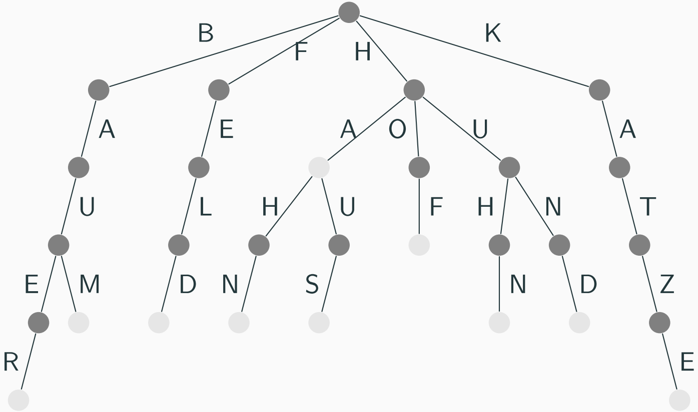
Tries



Tries



Tries



Tries

```
struct node {
    node* children[26];
    bool is_end;

    node() {
        memset(children, 0, sizeof(children));
        is_end = false;
    }
};
```

Tries

```
void insert(node* nd, char *s) {
    if (*s) {
        if (!nd->children[*s - 'a'])
            nd->children[*s - 'a'] = new node();

        insert(nd->children[*s - 'a'], s + 1);
    } else {
        nd->is_end = true;
    }
}
```

Tries

```
bool contains(node* nd, char *s) {
    if (*s) {
        if (!nd->children[*s - 'a'])
            return false;

        return contains(nd->children[*s - 'a'], s + 1);
    } else {
        return nd->is_end;
    }
}
```

Tries

```
node *trie = new node();

insert(trie, "banani");

if (contains(trie, "banani")) {
    // ...
}
```

Tries

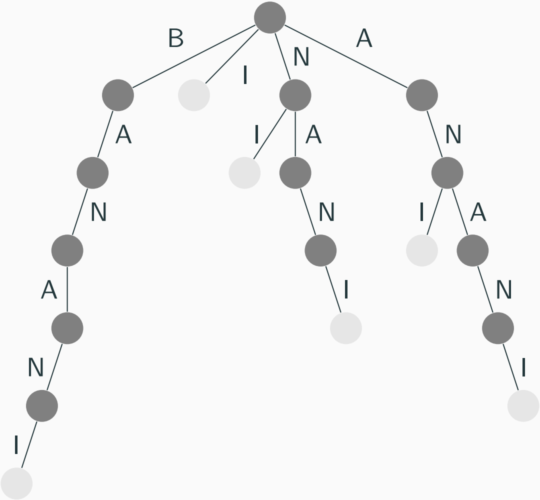
- Time complexity?

- Let k be the length of the string we're inserting/looking for
- Lookup and insertion are both $O(k)$

Suffix tries

- Say we're dealing with some string S of length n
- Let's insert all suffixes of S into a trie
- $S =$ `banani`
  - `insert(trie, "banani");`
  - `insert(trie, "anani");`
  - `insert(trie, "nani");`
  - `insert(trie, "ani");`
  - `insert(trie, "ni");`
  - `insert(trie, "i");`

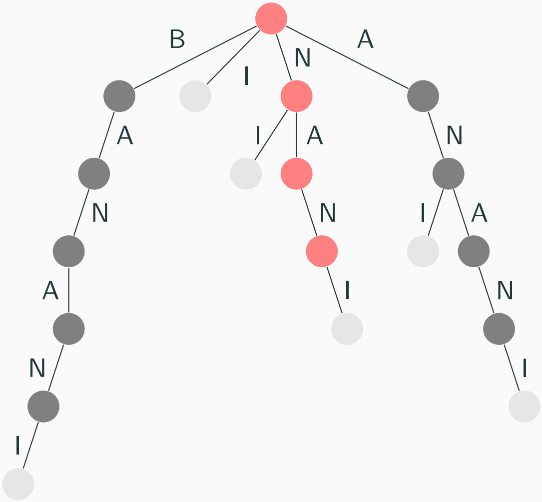
Suffix tries



Suffix tries

- There are a lot of cool things we can do with suffix tries
- Example: String matching
- If a string T is a substring in S, then (obviously) it has to start at some suffix of S
- So we can simply look for T in the suffix trie of S, ignoring whether the last node is an end node or not
- This is just $O(m)$...

Suffix tries



Suffix tries

- String matching is fast if we have the suffix trie for S

- But what is the time complexity of suffix trie construction?
- There are n suffixes, and it takes $O(n)$ to insert each of them
- So $O(n^2)$, which is pretty slow

- Can we do better?
- There can be up to n^2 nodes in the graph, so this is actually optimal...

Suffix trees

- There exists a compressed version of a suffix trie, called a suffix tree

- It can be constructed in $O(n)$, and has all the features that suffix tries have
- But the $O(n)$ construction algorithm is pretty complex, a big disadvantage for us

Suffix arrays

- A variation of the previous structures
- Can do everything the other structures can do, with a small overhead
- Can be constructed pretty quickly with relatively simple code

Suffix arrays

- Take all the suffixes of S

```
banani
anani
nani
ani
ni
i
```

- and sort them

```
anani
ani
banani
i
nani
ni
```

Suffix arrays

- We can use this array to do everything that suffix tries can do
- Like string matching

Suffix arrays

- Let's look for `nan`

```
anani
ani
banani
i
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The first letter in the string has to be `n`, so we can binary search for the range of strings starting with `n`

```
anani
ani
banani
i
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The first letter in the string has to be `n`, so we can binary search for the range of strings starting with `n`

```
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The second letter in the string has to be `a`, so we can binary search for the range of strings that have `a` as the second letter

```
nani
ni
```

Suffix arrays

- Let's look for `nan`
- The second letter in the string has to be `a`, so we can binary search for the range of strings that have `a` as the second letter

`nani`

Suffix arrays

- Let's look for `nan`
- The third letter in the string has to be `n`, so we can binary search for the range of strings that have `n` as the third letter

```
nani
```

Suffix arrays

- Let's look for `nan`
- The third letter in the string has to be `n`, so we can binary search for the range of strings that have `n` as the third letter

```
nani
```

Suffix arrays

- Let's look for `nan`
- The third letter in the string has to be `n`, so we can binary search for the range of strings that have `n` as the third letter

```
nani
```

- If there is at least one string left, we have a match

Suffix arrays

- Time complexity?
- For each letter in T, we do two binary searches on the n suffixes to find the new range
- Time complexity is $O(m \times \log n)$
- A bit slower than doing it with a suffix trie, but still not bad

Suffix arrays

- But how do we construct a suffix array for a string?
- A simple `sort(suffixes)` is $O(n^2 \log(n))$, because comparing two suffixes is $O(n)$
- And we still have the same problem as with suffix tries, there are almost n^2 characters if we store all suffixes

Suffix arrays

- The second problem is easy to fix
- Just store the indices of the suffixes

```
anani
ani
banani
i
nani
ni
```

- becomes

```
1: anani
3: ani
0: banani
5: i
2: nani
4: ni
```

Suffix arrays

- What about the construction?
- In short, we
  - sort all suffixes by only looking at the first letter
  - sort all suffixes by only looking at the first 2 letters
  - sort all suffixes by only looking at the first 4 letters
  - sort all suffixes by only looking at the first 8 letters
  - ...
  - sort all suffixes by only looking at the first 2^i letters
  - ...

- If we use an $O(n \log n)$ sorting algorithm, this is $O(n \log^2 n)$
- We can also use an $O(n)$ sorting algorithm, since all sorted values are between 0 and n, bringing it down to $O(n \log n)$

Suffix arrays

```
struct suffix_array {
    struct entry {
        pair<int, int> nr;
        int p;

        bool operator <(const entry &other) const {
            return nr < other.nr;
        }
    };


    string s;
    int n;
    vector<vector<int> > P;
    vector<entry> L;
    vi idx;

    // constructor
};
```

Suffix arrays

```
suffix_array(string _s) : s(_s), n(s.size()) {
    L = vector<entry>(n);
    P.push_back(vi(n));
    idx = vi(n);

    for (int i = 0; i < n; i++) {
        P[0][i] = s[i];
    }

    for (int stp = 1, cnt = 1; (cnt >> 1) < n; stp++, cnt <<= 1) {
        P.push_back(vi(n));
        for (int i = 0; i < n; i++) {
            L[i].p = i;
            L[i].nr = make_pair(P[stp - 1][i], i + cnt < n ? P[stp - 1][i + cnt] : -1);
        }

        sort(L.begin(), L.end());
        for (int i = 0; i < n; i++) {
            if (i > 0 && L[i].nr == L[i - 1].nr) {
                P[stp][L[i].p] = P[stp][L[i - 1].p];
            } else {
                P[stp][L[i].p] = i;
            }
        }
    }

    for (int i = 0; i < n; i++) {
        idx[P[P.size() - 1][i]] = i;
    }
}
```

Suffix arrays

- There is also one other useful operation on suffix arrays
- Finding the longest common prefix (lcp) of two suffixes of S

```
1: anani
3: ani
0: banani
5: i
2: nani
4: ni
```

- `lcp(1,3) = 2`
- `lcp(2,1) = 0`

- This function can be implemented in $O(\log n)$ by using intermediate results from the suffix array construction

Suffix arrays

```
int lcp(int x, int y) {
    int res = 0;
    if (x == y) return n - x;
    for (int k = P.size() - 1; k >= 0 && x < n && y < n; k--) {
        if (P[k][x] == P[k][y]) {
            x += 1 << k;
            y += 1 << k;
            res += 1 << k;
        }
    }
    return res;
}
```

Longest common substring

- Given two strings S and T, find their longest common substring

- `S = banani`
- `T = kanina`

- Their longest common substring is `ani`

- *see example*